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CONTRACTION GROUPS AND SCALES OF AUTOMORPHISMS OF TOTALLY DISCONNECTED LOCALLY COMPACT GROUPS

BY

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ABSTRACT

We study contraction groups for automorphisms of totally disconnected locally compact groups using the scale of the automorphism as a tool. The contraction group is shown to be unbounded when the inverse automorphism has non-trivial scale and this scale is shown to be the inverse value of the modular function on the closure of the contraction group at the automorphism. The closure of the contraction group is represented as acting on a homogenous tree and closed contraction groups are characterised.

1. Introduction

Interest in contraction groups and related concepts has been stimulated by applications in the theory of probability measures and random walks on groups and in representation theory.

In representation theory, contraction groups bring about the Mautner phenomenon. This is manifest in Wang's examination [Wan84] of the phenomenon in p-adic Lie groups, though more in the background in Moore's treatment of the Lie group case [Moo80]. To our knowledge the first to define and study contraction groups (in a slightly more general context than ours, see [MR76]) was Müller-Römer, who did so to study a representation theoretic question as well (the Wiener and Tauber property of a group algebra).

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When studying semistable convolution semigroups of probability measures on locally compact groups, contraction groups arise naturally [HS88, DS91]. For this reason a lot of preliminary work on contraction groups was done by Siebert, and most of the known results are either derived or referenced in [Sie86]. Proposition 4.2 in [Sie86] reduces the study of locally compact contraction groups to a separate study of the connected and totally disconnected cases. The connected case is covered by Corollary 2.4 in *loc. cit.* In some totally disconnected groups the contraction groups are closed and these are studied in [Sie89], see also Remark 3.33 below. We contribute to the full picture by treating general totally disconnected groups.

Just as Lie techniques are used to study contraction groups in the connected case, so the notions of scale and tidy subgroup (introduced in [Wil94]) are useful in the totally disconnected case. The relevant properties of tidy subgroups are summarized in Section 2. Their connection to contraction subgroups and related concepts is explored in Section 3. The basic properties are collected in Subsection 3.1. This subsection culminates in our main result, Theorem 3.8, whose proof rests on the assumption that the group is metrizable.

Therefore, in the following sections we assume that all groups considered are totally disconnected locally compact metric unless explicitly stated otherwise.

From the reinterpretation of the scale function as the modular function restricted to various subgroups (Proposition 3.21) we infer that contraction groups of automorphisms whose inverse has non-trivial scale are unbounded (Corollary 3.24). We succeed in Proposition 3.32 to characterize closed contraction groups. Theorem 4.2 in Section 4 then represents contraction groups as groups of automorphisms of a homogeneous tree.

We stick to the following conventions: 0 is a natural number. The relations $\subset$, $\lhd$ etc. always imply strict inclusion. Any automorphism of a topological group will be assumed to be a homeomorphism. By "X is stable under α" we mean $\alpha(X) = X$ whereas "X is invariant under α" means $\alpha(X) \subseteq X$. The modular function Δ_G of a locally compact topological group G is defined by the equation $\mu(\alpha(M)) = \Delta_G(\alpha)\mu(M)$ where μ is a left Haar measure on G. We use e for the unit element of a group and 1 for the trivial group. The function $|\cdot|$ stands for the absolute value on complex or p-adic numbers, as the case may be.

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2. Basic facts about the scale function

Tidy subgroups for an automorphism α of a totally disconnected locally compact group G provide a local description of α. For a subgroup V of G define subgroups V_+, V_- of V and V_{++}, V_{--} of G by

$$V_\pm = \bigcap_{n\geq 0} \alpha^{\pm n}(V) \quad \text{and} \quad V_{\pm\pm} = \bigcup_{n\geq 0} \alpha^{\pm n}(V_\pm).$$

The automorphism α magnifies V_+ and shrinks V_-, that is, $\alpha(V_+) \geqslant V_+$ and $\alpha^{-1}(V_-) \geqslant V_-$. We let V_0 denote the "neutral" part:

$$V_0 := V_+ \cap V_- = \bigcap_{k\in\mathbb{Z}} \alpha^k(V) = \bigcap_{j\in\mathbb{N}} \alpha^j(V_-) = \bigcap_{j\in\mathbb{N}} \alpha^{-j}(V_+).$$

It is stable under α.

Definition 2.1: Let G be a totally disconnected locally compact group and α an automorphism of G. A compact-open subgroup V is called **tidy for α in G** (or **tidy** for short if α is understood) if it satisfies

(T1) $V = V_+V_-$ $(= V_-V_+)$,

(T2) V_{--} (and V_{++}) are closed.

The integer

$$s_G(\alpha) := |\alpha(V_+) : V_+| = |\alpha(V) : V \cap \alpha(V)|$$

is called the **scale of α**. The function $s_G\colon G \to \mathbb{N}$ obtained by restricting attention to inner automorphisms will be called the **scale function** of G.

The scale of an automorphism is well defined by Theorem 2 in [Wil94]. Observe that a subgroup V which is tidy for α will be tidy for α^{-1} as well.

Tidy subgroups exist. They may be found by running the following algorithm.

ALGORITHM 2.2 (cf. proof of Theorem 3.1 in [Wil01]):

[0] Choose a compact open subgroup $O \leqslant G$.

[1] Let ${}^kO := \bigcap_{i=0}^k \alpha^i(O)$. We have ${}^kO_+ = O_+$ and ${}^kO_- = \alpha^k(O_-)$. For some $n \in \mathbb{N}$ (hence for all $n' \geq n$) the group ${}^nO$ satisfies (T1). Put $O' := {}^nO$.

[2] Let $\mathcal{L} := \{x \in G\colon \alpha^i(x) \in O' \text{ for almost all } i \in \mathbb{Z}\}$ and $L := \overline{\mathcal{L}}$.

[3] Form $O^* := \{x \in O'\colon lxl^{-1} \in O'L\ \forall l \in L\}$ and define $O'' := O^*L$. The group O'' is tidy and we output O''.

Analyzing the above algorithm (cf. [Wil01, Theorem 3.1]) one arrives at the alternative characterization of the scale function as a minimal distortion value:

$$s_G(\alpha) = \min\{|\alpha(O) : O \cap \alpha(O)|\colon O \text{ a compact, open subgroup}\}.$$

The following simple criterion ensuring tidiness will be used repeatedly in the sequel.

Example 2.3: A compact, open subgroup V satisfying $V = V_-$ is tidy. ∎

It follows immediately from the definition of the scale that α has a tidy subgroup satisfying this criterion if and only if $s(\alpha) = 1$.

3. Contraction groups and parabolics

Let α be an automorphism of a locally compact group G. This section links the subgroups tidy for α to two other subgroups having global dynamical definitions in terms of α.

3.1. DEFINITIONS AND BASIC PROPERTIES. We define the parabolic group and the contraction group of α and begin to elucidate links between the various groups. Tidy subgroup methods show that the parabolic group is closed. They are also used in the proof of Theorem 3.8, which is fundamental for all of our results on contraction groups.

Remark 3.1: If V is a compact, open subgroup of G, then we have

$$V_- \subseteq P_\alpha := \{x \in G\colon \{\alpha^n(x)\colon n \in \mathbb{N}\} \text{ is bounded}\}$$

and $V_- = V \cap P_\alpha$ if V is tidy for α (see Lemma 9 in [Wil94]).

The group P_α is closed (Proposition 3 parts (iii) and (ii) in [Wil94] show this) and obviously contains the group

$$U_\alpha := \{x \in G\colon \alpha^n(x) \longrightarrow_{n\to\infty} e\},$$

which need not be closed (see Example 3.13(2)).

Notation 3.2: We call P_α and U_α respectively the **parabolic** subgroup and the **contraction** group associated to α. We also let $M_\alpha := P_\alpha \cap P_{\alpha^{-1}}$ and call it the **Levi factor** attached to α. The Levi factor is the set of all elements of the ambient group whose $\langle\alpha\rangle$-orbit is bounded. If the automorphism α is inner and is conjugation by g, we relax notation and write P_g, U_g and M_g.

The term 'parabolic group' is suggested by Example 3.13 to follow. The name 'contraction group', however, is standard; see [MR76, Wan84, Sie86, Sie89]. Results about contraction groups, notably in the case where U_α is closed, may be found in these papers. The next result is one such. We shall use it frequently.

LEMMA 3.3 ([Wan84], Proposition 2.1): *Let G be a locally compact group and let α be an automorphism of G such that $U_\alpha = G$. Then α is compactly contractive, that is, for any compact subset C of G and any neighborhood O of e we have $\alpha^n(C) \subseteq O$ for all $n \geq N(C, O)$.*

Since U_α is α-stable, this will imply that whenever the ambient group is locally compact and U_α is closed, the restriction of α to U_α is compactly contractive.

Since α shrinks V_-, one feels that there should be an even closer connection between V_- and U_α than the one between V_- and P_α displayed by Remark 3.1. Before establishing this, we note **some elementary properties of the groups U_α, P_α and M_α**:

$$U_{\alpha^n} = U_\alpha \quad \text{and} \quad P_{\alpha^n} = P_\alpha \quad \text{for all } n \in \mathbb{N} \setminus \{0\};$$
$$U_{\mathrm{id}} = 1 \quad \text{and} \quad P_{\mathrm{id}} = G, \quad \beta(U_\alpha) = U_{\beta\alpha\beta^{-1}}, \quad \beta(P_\alpha) = P_{\beta\alpha\beta^{-1}}.$$

Further, as is plain from the definitions, when computing the contraction groups and parabolics inside a subgroup (stable under the automorphism in question) we get the intersections of the contraction groups and parabolics in the ambient group respectively with the subgroup.

Obviously $M_\alpha = M_{\alpha^{-1}}$ and $V_0 \leqslant V \cap M_\alpha$ for every compact open subgroup with equality if V is tidy thanks to Remark 3.1.

PROPOSITION 3.4: *Let G be a locally compact group and let α be an automorphism. Then U_α is normal in P_α, hence $U_\alpha M_\alpha \leqslant P_\alpha$.*

Proof: Let $x \in P_\alpha$, $u \in U_\alpha$ be given. By definition of P_α the set $\overline{\{\alpha^n(x) \colon n \in \mathbb{N}\}}$, name it K, is compact. Given an open neighborhood O of e, choose an open neighborhood O' of e satisfying $\bigcup_{k\in K} kO'k^{-1} \subseteq O$ ([HR79, II.4.9]). Then from $\alpha^n(u) \in O'$ for all $n \geq n_0$ we infer $\alpha^n(xux^{-1}) = \alpha^n(x)\alpha^n(u)\alpha^n(x)^{-1} \in \bigcup_{k\in K} kO'k^{-1} \subseteq O$, proving that indeed $xux^{-1} \in U_\alpha$. ∎

Theorem II.4.9 from [HR79] can also be used to prove the first claim in the following result, which is useful in computations.

LEMMA 3.5: *Let G be a locally compact group and let $d, v \in G$ be such that $dv = vd$ and $\langle v \rangle$ is bounded. Then $U_{dv} = U_d$ and $P_{dv} = P_d$.* ∎

We note the following simple consequence.

COROLLARY 3.6: *Let G be a locally compact group and $g \in G$ be given. Then either of $U_g \neq 1$ or $P_g \neq G$ implies that $\overline{\langle g \rangle} = \langle g \rangle$ is infinite cyclic.* ∎

Next, we investigate the behavior of contraction groups and parabolics under quotient maps. Recall that a continuous surjection $f\colon X \to Y$ between topological spaces is an **identification**, respectively a **perfect map**, if Y carries the quotient topology with respect to f, respectively if it is a closed map with all fibres compact. If f is a homomorphism of topological groups it is an identification iff it is a continuous open surjection. It is perfect, if in addition its kernel is compact.

PROPOSITION 3.7: *Let $p\colon G \to \overline{G}$ be a homomorphism of totally disconnected locally compact metric groups which is an identification. Let α be an automorphism of G leaving the kernel of p stable and thus inducing an automorphism $\overline{\alpha}$ of $\overline{G}$. Then $p(U_\alpha) = U_{\overline{\alpha}}$.*

Proposition 3.7 is obtained as a corollary of the next result taking $H := \ker p$. We first introduce some notation. Let H be a subset of the topological group G. Call a sequence of elements $(x_n)_{n\in\mathbb{N}}$ in G convergent to e modulo H, if for any e-neighbourhood V there is an integer N_V such that VHV contains all terms of the subsequence $(x_n)_{n\geq N_V}$ and write $\lim_{n\in\mathbb{N}} x_n = e \bmod H$ in this case. Let $U_{\alpha/H} := \{x \in G\colon \lim_{n\in\mathbb{N}} \alpha^n(x) = e \bmod H\}$. If H is a neutral subgroup of G, that is, if for each e-neighborhood V there is an e-neighbourhood W such that $WH \subseteq HV$, in particular if H is compact, then $U_{\alpha/H}$ is a group.

THEOREM 3.8: *Let G be a totally disconnected locally compact metric group, α an automorphism of G and H an α-stable closed subgroup of G. Then $U_{\alpha/H} = U_\alpha H$.*

The criterion provided by the following lemma is useful in the proof.

LEMMA 3.9: *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a locally compact group G and let H be a subset of G. Then the following statements hold.*

(1) *Let $\{x_n\colon n \in \mathbb{N}\}$ be bounded. If H contains each accumulation point of $(x_n)_{n\in\mathbb{N}}$ then $(x_n)_{n\in\mathbb{N}}$ converges to e modulo H.*
(2) *If $(x_n)_{n\in\mathbb{N}}$ converges to e modulo H, then each accumulation point of $(x_n)_{n\in\mathbb{N}}$ is contained in $\overline{H}$.*
(3) *If $\{x_n\colon n \in \mathbb{N}\}$ is bounded and H is closed then $(x_n)_{n\in\mathbb{N}}$ converges to e modulo H iff H contains each accumulation point of $(x_n)_{n\in\mathbb{N}}$.*

Proof of Lemma 3.9: (2) follows from the definitions, while (3) is implied by the statements (1) and (2). It remains to prove (1).

We argue by contradiction. Let W be a neighborhood of H such that its complement contains infinitely many elements of the set $\{x_n\colon n \in \mathbb{N}\}$. Since $\{x_n\colon n \in \mathbb{N}\}$ is bounded, the subsequence $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ formed by the elements not in W has an accumulation point, which is in H by assumption. Then W must contain infinitely many elements of the set $\{x_{n_i}\colon i \in \mathbb{N}\}$ contradicting its definition. ∎

Proof of Theorem 3.8: The right hand side is contained in the left hand side. To show the opposite inclusion, we need to show that whenever $\alpha^n(x)$ converges to e modulo H, then there is an element h of H, such that $\alpha^n(xh)$ converges to e.

Since G is metric, there is a decreasing sequence $(O^{(i)})_{i=1}^{\infty}$ of compact open subgroups of G with trivial intersection. Put $O^{(0)} := G$. We use induction on $i \in \mathbb{N}$ to show that there are sequences of elements $(y_i)_{i\in\mathbb{N}}$ in G and natural numbers $(N_i)_{i\in\mathbb{N}}$ such that

(1) $y_0 = x$, $y_{i+1} \in y_i(O_0^{(i)} \cap H)$ for all $i \in \mathbb{N}$,
(2) $\alpha^n(y_i) \in O^{(i)}$ for all $n \geq N_i$,
(3) $\lim_{n\in\mathbb{N}} \alpha^n(y_i) = e \bmod O_0^{(i)} \cap H$.

Putting $y_0 := x$ and $N_0 := 0$ provides a basis for the induction. For the induction step we will use the following lemma.

LEMMA 3.10: *Let G be a totally disconnected locally compact group, α an automorphism of G, H an α-stable closed subgroup of G and x an element of $U_{\alpha/H}$. Then, for any compact open subgroup O of G there is an element h in H and a natural number N such that $\alpha^n(xh)$ is contained in O for each $n \geq N$. The sequence $(\alpha^n(xh))_{n\in\mathbb{N}}$ converges to e modulo $O_0 \cap H$.*

Proof of Lemma 3.10: Applying step 1 of the Algorithm 2.2 to $O \cap H$, we may assume that the intersection of O with H satisfies property (T1) with respect to α, hence $O \cap H = (O \cap H)_-(O \cap H)_+$. Using continuity of α, choose a compact open subgroup V of O such that $\alpha(V) \subseteq O$. Then $\alpha(V(O \cap H)) \subseteq O\alpha(O \cap H)$. Let N be such that $\alpha^n(x)$ is contained in VH for all $n \geq N$.

Choose an element h_0 in H such that $\alpha^N(xh_0) \in V$. We will complete h_0 recursively to a sequence (h_i) of elements in $h_0(O \cap H)$ such that

$$\alpha^{N+j}(xh_i) \in V(O \cap H) \quad \text{for } 0 \leq j \leq i.$$

The recursion starts with h_0. Suppose then, that h_0 up to h_k have already been constructed to satisfy this condition. Then, using

$$\alpha(O \cap H) \subseteq (O \cap H)\alpha((O \cap H)_+)$$

we obtain

$$\alpha^{N+k+1}(xh_k) \in \alpha(V(O \cap H)) \subseteq O\alpha(O \cap H) \subseteq O\alpha((O \cap H)_+).$$

Choose l_{k+1} in $(O \cap H)_+$ such that $\alpha(\alpha^{N+k}(xh_k)l_{k+1}^{-1}) \in O$ and put $h_{k+1} := h_k\alpha^{-(N+k)}(l_{k+1}^{-1})$. Then the element h_{k+1} is in $h_0(O \cap H)$, and $\alpha^{N+k+1}(xh_{k+1}) \in O$. Further, since xh_k is in the same H-coset as x, $\alpha^{N+k+1}(xh_{k+1}) \in VH$, by the definition of N.

Hence $\alpha^{N+k+1}(xh_{k+1}) \in O \cap VH = V(O \cap H)$. For all natural numbers j less than $k+1$ we have $\alpha^{N+j}(xh_{k+1}) = \alpha^{N+j}(xh_k)\alpha^{j-k}(l_{k+1}^{-1}) \in V(O \cap H)$ as well, showing the existence of our announced sequence.

The sequence $(xh_i)_{i\in\mathbb{N}}$ is bounded, hence has an accumulation point xh in xH. Then $\alpha^{N+j}(xh)$ is in $V(O \cap H)$ for any natural number j because $V(O \cap H)$ is closed. In particular, it is in O, showing the first claim.

By the first claim, which we already proved, $\{\alpha^n(xh)\colon n \in \mathbb{N}\}$ is bounded. Continuity of α and α^{-1} imply that the set of accumulation points of $(\alpha^n(xh))_{n\in\mathbb{N}}$ is an α-stable subset of O. Hence each of these accumulation points belongs to O_0. On the other hand, $(\alpha^n(x))_{n\in\mathbb{N}}$ hence $(\alpha^n(xh))_{n\in\mathbb{N}}$ converges to e modulo H. This can be reformulated using part 3 of Lemma 3.9 to read that each accumulation point of $(\alpha^n(xh))_{n\in\mathbb{N}}$ belongs to H. We conclude that each accumulation point of $(\alpha^n(xh))_{n\in\mathbb{N}}$ belongs to $H \cap O_0$. Applying part 3 of Lemma 3.9 once more we see that the sequence $(\alpha^n(xh))_{n\in\mathbb{N}}$ converges to e modulo the compact α-stable subgroup $H \cap O_0$ of H, and we have established the second claim. The lemma is proved. ∎

Returning to the proof of the theorem, assume that the induction hypothesis has been established for i. We apply the lemma with $O_0^{(i)} \cap H$ in place of H, $O^{(i+1)}$ in place of O and y_i in place of x. We deduce that there is an element $h_i \in O_0^{(i)} \cap H$ and an integer N_{i+1} such that $\alpha^n(y_ih_i)$ is contained in $O^{(i+1)}$ for each $n \geq N_{i+1}$. Furthermore, the sequence $(\alpha^n(y_ih_i))_{n\in\mathbb{N}}$ converges to e modulo $O_0^{(i+1)} \cap H$. Putting $y_{i+1} := y_ih_i$, this gives the induction statement for $i+1$ proving that the statement holds for all positive integers.

Since $y_{i+1} \in y_i(O_0^{(i)} \cap H)$ and $O^{(i+1)} \leqslant O^{(i)}$, $(y_i(O_0^{(i)} \cap H))_{i=1}^{\infty}$ is a decreasing sequence of compact sets and $\bigcap_{i=1}^{\infty} y_i(O_0^{(i)} \cap H)$ is a single point because $\bigcap_{i=1}^{\infty} O^{(i)}$ is trivial. Let $\{y\} = \bigcap_{i=1}^{\infty} y_i(O_0^{(i)} \cap H)$. Then for every n

$$\alpha^n(y) \in \alpha^n(y_i(O_0^{(i)} \cap H)) = \alpha^n(y_i)\alpha^n(O_0^{(i)} \cap H).$$

Since $O_0^{(i)} \cap H$ is α-stable, this set equals $\alpha^n(y_i)(O_0^{(i)} \cap H)$, which is contained in $O_0^{(i)}$ for every $n \geq N_i$. Hence $\alpha^n(y)$ converges to e as n tends to ∞. Since

$y_0 = x$, $y_{i+1} \in y_i H$ and H is closed, $x^{-1}y \in H$ and so the proof is completed by setting $h := x^{-1}y$. ∎

Groups of the form $U_{\alpha/H}$ with H compact arise naturally when studying (semi-)stable convolution semigroups of probability measures on the ambient group; see [HS88, DS91]. Theorem 3.8 above generalizes Theorem 2.4 in [DS91]. Theorem 3.1 in [HS88] covers the case where H is a compact subgroup of a Lie group. We pose the question whether Theorem 3.8 can be generalised to include the non-metric case.

The analogue of the above Theorem for parabolic groups and their Levi factors does not hold. Indeed, there are discrete counterexamples.

Example 3.11: Let G be a finitely generated discrete group such that $G \neq [G,G] = Z(G)$ and $G/[G,G]$ is torsion free. For example, take G to be the group of integral strict upper triangular matrices of rank 3. Take $L := [G,G]$ and let g be some element of G. Since G/L is abelian, the parabolic group (and its Levi factor) attached to $gL \in G/L$ is the whole group. We will show that P_g surjects onto P_{gL} only in the trivial case $g \in L$, providing the desired counterexample.

The assumption that P_g surjects onto $P_{gL} = G$ implies that $G = P_g L = P_g Z(G) = P_g$. Hence every element of G has only a finite number, $m(x)$ say, of $\langle g \rangle$-conjugates and we infer that $[g^{m(x)}, x] = e$. But G is finitely generated, thus there is a positive M such that $[g^M, x] = e$ holds for all x in G. In other words, g^M is in $Z(G)$. Since $G/Z(G) = G/[G,G]$ is torsion free, this implies that g belongs to the center of G, that is, it belongs to L as claimed.

Though Proposition 3.7 does not generalize to parabolic groups and Levi factors, the following weaker result is obvious.

PROPOSITION 3.12: *Let p: $G \to \overline{G}$ be a perfect homomorphism of locally compact groups and let α be an automorphism of G leaving the kernel of p stable and thus inducing an automorphism $\overline{\alpha}$ of $\overline{G}$. Then $p^{-1}(P_{\overline{\alpha}}) = P_\alpha$ and $p^{-1}(M_{\overline{\alpha}}) = M_\alpha$. In particular, $p(P_\alpha) = P_{\overline{\alpha}}$ and $p(M_\alpha) = M_{\overline{\alpha}}$* ∎

We conclude the subsection with some examples illustrating these concepts.

Example 3.13: The following examples present different types of behavior which can occur. The first and second of these should provide the reader with geometrical intuition on what is going on.

(1) Let k be a locally compact, totally disconnected field, e.g. the p-adic numbers $\mathbb{Q}_p$. Let G be $SL_n(k)$, equipped with the subspace topology in k^{n^2}. Then G is a totally disconnected locally compact group.

In the case where $G = SL_n(\mathbb{Q}_p)$, it follows from Lemma 3.5 that, when computing U_g and P_g, we may suppose that g is semisimple. That means that, after conjugation, we may assume that g is diagonal over some finite extension field of $\mathbb{Q}_p$ (also see Proposition 3.23). We may further assume that the valuations of the diagonal entries are in decreasing order. It is easy to compute U_g, P_g and M_g with this normalization and one finds that U_g is a closed (algebraic) normal subgroup of P_g consisting of unipotent matrices, and that P_g is the semidirect product of M_g and U_g. If U_g is bounded, we have $U_g = 1$. If all eigenvalues have distinct absolute value, M_g, P_g and U_g will be the groups of diagonal, upper and strictly upper triangular matrices, respectively. If $\mathbf{G}$ is any semisimple group, then essentially the same results hold for the group $\mathbf{G}(k)$: P_g is the group of rational points of a k-parabolic subgroup; U_g is the group of rational points of its unipotent radical; and M_g is the group of rational points of the centralizer of the unique k-split torus contained in $P_g \cap P_{g^{-1}}$. (This is essentially contained in Lemma 2 of [Pra82]. One should note that the hypothesis that $\mathbf{G}$ is almost k-simple is not used in its proof and that the hypothesis on the eigenvalues of Ad(g) is only needed to ensure that P_g is a proper subgroup.)

The scale function was computed for general and special linear groups over local skew fields (and some other linear groups) by Glöckner in [Glö98a]. This list includes the group $SL_n(\mathbb{Q}_p)$ discussed above. In that group the subgroup $\mathrm{id} + M_n(p\mathbb{Z}_p)$ is shown to be tidy for diagonal g. It turns out that $s(g)$ is the product of the absolute values of those eigenvalues of Ad(g) which have absolute value greater than or equal to 1. The scale function is computed for connected semisimple algebraic groups over arbitrary non-Archimedean fields in Proposition 3.23. If the characteristic of the field is 0 one can alternatively use Lie methods to compute the scale function as done in [Glö98b].

(2) Let T be the homogeneous tree of degree $q+1$. Taking fixators of finite sets of vertices as basic neighborhoods of the identity induces a totally disconnected locally compact group topology on $\mathrm{Aut}(T)$. Each automorphism of T either has a fixed point, not necessarily a vertex (elliptic case) or a stable line, which is unique and is called the axis of g (hyperbolic case).

An elliptic element g is topologically periodic, hence has $U_g = 1$, $P_g = \mathrm{Aut}(T) = M_g$ and trivial scale. The stabilizer of any point fixed by g is a tidy subgroup since it contains g. If g is hyperbolic, then it is easily shown that P_g is the stabilizer of the repelling end ϵ_- of g. An automorphism x is in U_g if and only if for each $r > 0$ there is a point $p(x, r)$ on the axis of g such that

all points within distance r of the ray $[p(x,r),\epsilon_-)$ are fixed by x. In this case, U_g is unbounded but not closed (the closure of U_g is the set of elliptic elements fixing ϵ_-). The group M_g being the fixator of the two ends fixed by g, we easily see that $M_g \cap U_g$ is nontrivial. The scale of a hyperbolic element is $q^{l(g)}$, where $l(g)$ is the length of the translation induced by g on its axis. The fixator of a segment of length at least one on the axis of g is tidy for g (one can show using Lemma 3.31(3) that every tidy subgroup for g is essentially of that form). This example was treated in detail in section 3 of [Wil94].

(3) Let H be a totally disconnected locally compact group and O a compact, open subgroup. For example, we may take both H and O equal to a finite group F carrying the discrete topology. The shift map $i \mapsto i+1$ on $\mathbb{Z}$ induces an automorphism σ of the restricted product $G := \prod_{i\in\mathbb{Z}} H|O$ (for definition see editorial comment after entry Adèle on page 43 in [Haz88]). Any compact subset stable under the shift σ is contained in $O^{\mathbb{Z}} = \prod_{i\in\mathbb{Z}} O$ and any open stable set contains $O^{\mathbb{Z}}$. Therefore, $O^{\mathbb{Z}}$ is the unique σ-stable compact, open subgroup of G, and thus the only subgroup tidy for σ. As a corollary, σ has scale 1. The parabolic subgroup P_σ and Levi-factor M_σ are both equal to $O^{\mathbb{Z}}$. The contraction group U_σ is the subgroup of all $(x_i)_{i\in\mathbb{Z}} \in O^{\mathbb{Z}}$ such that $x_i \longrightarrow_{i\to-\infty} e$. This is a nontrivial and bounded group which is not closed.

3.2. THE DIFFERENCE BETWEEN SHRINKING AND CONTRACTING. From now on we assume that the ambient group G is totally disconnected and **metrizable** unless explicitly stated otherwise. We establish the connection between that part, V_-, of a compact, open subgroup V on which α is shrinking and the corresponding contraction group U_α. The notion of contractivity is stronger than that of shrinking, as the first lemma demonstrates.

LEMMA 3.14: *Let V be a compact, open subgroup of a totally disconnected locally compact group. Then $U_\alpha \leqslant V_{--}$.*

Proof: Take $v \in U_\alpha$. Then there is an N such that $\alpha^n(v)$ belongs to (the e-neighborhood) V for every $n \geq N$. Thus $\alpha^N(v) \in V_-$ and $v \in V_{--}$. ∎

Since $V_- \subseteq P_\alpha$ (see Remark 3.1), the following result is an immediate consequence of the lemma and Proposition 3.4.

COROLLARY 3.15: *Let V be a compact, open subgroup of a totally disconnected locally compact group. Then $U_\alpha \trianglelefteq V_{--}$, $U_\alpha \cap V_- \trianglelefteq V_-$ and $U_\alpha \cap V_0 \trianglelefteq V_0$.* ∎

We now make the interdependence between U_α and V_{--} more precise. Clearly, $U_\alpha V_0 \leqslant V_{--}$ and Lemma 3.3 implies that $V_{--} \subseteq U_\alpha$ only when $V_0 = 1$. In fact,

V_0 is the 'difference' between V_{--} and U_α.

PROPOSITION 3.16: *For a compact, open subgroup V, $V_{--} = U_\alpha V_0$.*

Proof: We are left to show $V_{--} \subseteq U_\alpha V_0$. For each $n \in \mathbb{N}$ and $v \in V$ we have $\alpha^n(v) \in \bigcap_{j=-N}^{N} \alpha^j(V)$ for all sufficiently large n. Since $\bigcap_{j=-N}^{N} \alpha^j(V)$ is compact and decreases to V_0 with N, $(\alpha^n(v))_{n\in\mathbb{N}}$ converges to e modulo V_0 for all $v \in V_{--}$. The result follows from Theorem 3.8. ∎

The next result may be considered a global version of the above propositon.

COROLLARY 3.17: *$M_\alpha U_\alpha = P_\alpha$, in other words $U_{\alpha/M_\alpha} = P_\alpha$.*

Proof: Using Proposition 3.4 it suffices to show $P_\alpha \subseteq U_{\alpha/M_\alpha}$. Let v be in P_α. By definition of P_α the sequence $(\alpha^n(v))_{n\in\mathbb{N}}$ is bounded. Let w be an accumulation point, i.e. a limit of a subsequence. Then for any $m \in \mathbb{Z}$ the point $\alpha^m(w)$ as well is a limit of a subsequence and belongs to the compact set $\overline{\{\alpha^n(v)\colon n \in \mathbb{N}\}}$. This means $w \in M_\alpha$. Since the set $\{\alpha^n(v)\colon n \in \mathbb{N}\}$ is bounded and M_α is closed, we may apply part 3 of Lemma 3.9 to conclude that $v \in U_{\alpha/M_\alpha}$ as claimed. ∎

As might be expected, the topology of the quotient space $P_\alpha/\overline{U}_\alpha$ is induced from M_α.

LEMMA 3.18: *The natural homomorphism $M_\alpha \to P_\alpha/\overline{U}_\alpha$ is an identification.*

Proof: The homomorphism is continuous and surjective. To show that it is open, it suffices to find an open subgroup of M_α such that the restriction of the homomorphism to this subgroup is open.

Let V be a subgroup tidy for α in the ambient group. Then V_0 equals $V \cap M_\alpha$ and therefore is a compact open subgroup of M_α. We first claim that the image O of V_0 is open: The inverse image of O is $V_0\overline{U}_\alpha \leqslant P_\alpha$. The equations $V_0U_\alpha = V_{--} \geqslant V_- = V \cap P_\alpha$ show that V_0U_α is an open hence closed subgroup of P_α. It follows that $V_0\overline{U}_\alpha$ must equal V_0U_α and is therefore open. This proves that O is open as claimed.

By the open mapping theorem ([HR79, II.(5.29)]) the map $V_0 \to O$ is necessarily open. Since O was shown to be open, this implies that the map $V_0 \to P_\alpha/\overline{U}_\alpha$ is open. ∎

Lemma 3.29 and Corollary 3.27 will show that the kernel of the homomorphism $M_\alpha \to P_\alpha/\overline{U}_\alpha$ is compact. It then follows that this homomorphism is a perfect map.

3.3. REINTERPRETATION OF THE SCALE FUNCTION. We will investigate the links between contraction groups, parabolics and tidy subgroups further after giving alternative descriptions of the scale of an automorphism.

The next two lemmas are immediate consequences of Example 2.3 and Remark 3.1.

LEMMA 3.19: *Let V be a compact, open subgroup of a totally disconnected locally compact group tidy for the automorphism α. Then for all closed α-stable subgroups H of P_α*

$$(V \cap H)_- = V_- \cap H = V \cap P_\alpha \cap H = V \cap H$$

is tidy for α in H. ∎

LEMMA 3.20: *Let $V = V_-$ be a compact, open subgroup of the totally disconnected locally compact group H and let $N \trianglelefteq H$ be stable under the automorphism α of H. Then the image $q(V) \subseteq H/N$ under the canonical map satisfies*

$$q(V_-) \subseteq q(V)_- \subseteq q(V) = q(V_-),$$

and hence is tidy for the induced automorphism $\overline{\alpha}$: $H/N \to H/N$. ∎

The reason for the name 'scale' is that $s(\alpha)$ is the factor by which α scales up V_+ when V is tidy. Thus $s(\alpha)$ is just the value of the modular function at the restriction of α to V_{++}. The next result extends this interpretation of the scale.

PROPOSITION 3.21: *Let $N \trianglelefteq H$ be a α-stable closed subgroup of P_α and let V be tidy for α in the ambient group G. Then writing q for the canonical map $H \to H/N$ and $\overline{\alpha}$, respectively $\alpha_|$, for the induced automorphisms on H/N and N, we have*

(1) $s_H(\alpha^{-1}) = \Delta_H(\alpha^{-1})$,
(2) $s_H(\alpha^{-1}) = s_{H/N}(\overline{\alpha}^{-1}) s_N(\alpha_|^{-1})$,
(3) $s_G(\alpha^{-1}) = s_{P_\alpha}(\alpha^{-1}) = s_{V_{--}}(\alpha^{-1}) = s_{\overline{U}_\alpha}(\alpha^{-1})$.

Proof: (1) By Lemma 3.19 we have $(V \cap H)_- = V \cap H$ and this group is tidy for α in H. Hence $\Delta_H(\alpha^{-1}) = s_H(\alpha^{-1})$.

(2) By Lemma 3.20 we have $q(V \cap H)_- = q(V \cap H)$ and this group is tidy for $\overline{\alpha}$ in H/N. Hence $\Delta_{H/N}(\overline{\alpha}^{-1}) = s_{H/N}(\overline{\alpha}^{-1})$. Additionally, applying (1) with $H := N$ implies $\Delta_N(\alpha_|^{-1}) = s_N(\alpha_|^{-1})$.

Since the modular function satisfies an equation of the form to be proved, substituting the values of the scale function for those of the modular function in that equation proves (2).

(3) We have $s_G(\alpha^{-1}) = |\alpha^{-1}(V_-) : V_-| = s_{P_\alpha}(\alpha^{-1})$ since $V_- = V \cap P_\alpha = (V \cap P_\alpha)_-$. The same argument works when P_α is replaced by V_{--}. Now (2) applied with $H := P_\alpha$ and $N := \overline{U}_\alpha$ gives $s_{P_\alpha}(\alpha^{-1}) = s_{P_\alpha/\overline{U}_\alpha}(\overline{\alpha}^{-1}) s_{\overline{U}_\alpha}(\alpha_|^{-1})$, leaving us to prove $s_{P_\alpha/\overline{U}_\alpha}(\overline{\alpha}^{-1}) = 1$.

As the product of the scale of the restriction of an automorphism to a normal subgroup and the scale of the induced automorphism on the quotient always divides the scale of the automorphism by Proposition 4.7 in [Wil01], it is enough to show $s_{M_\alpha}(\alpha^{-1}) = 1$, since $P_\alpha/\overline{U}_\alpha$ is a quotient of M_α by Lemma 3.18. We compute

$$s_{M_\alpha}(\alpha^{-1}) = |\alpha^{-1}(M_\alpha \cap V)_- : (M_\alpha \cap V)_-|.$$

Observing that

$$M_\alpha \cap V = P_\alpha \cap P_{\alpha^{-1}} \cap V = V_+ \cap V_- = V_0,$$

this implies $s_{M_\alpha}(\alpha^{-1}) = |V_0 : V_0| = 1$, as had to be shown. ∎

Combination of (3) and (1) above implies that $s_G(g) = \Delta_{\overline{U}_{g^{-1}}}(g)$. This enables us to compute the scale function of the group of rational points of a semisimple algebraic group $\mathbf{G}$ over a local field of positive characteristic. We start with the following lemma.

LEMMA 3.22: *If α and β are two commuting automorphisms then*

(1) $s(\alpha\beta) \leq s(\alpha)s(\beta)$,

(2) $s(\beta) = 1 = s(\beta^{-1})$ *implies* $s(\alpha\beta) = s(\alpha)$.

Proof: As (2) follows from (1), it suffices to prove (1). We may suppose that α and β are inner. Using Theorem 3.4 in [Wil], choose a compact open subgroup V, which is tidy for α and β. The result follows from Proposition 7.2 in [GW02]. ∎

We now treat a slightly more general case than announced in Example 3.13.

PROPOSITION 3.23: *Let k be a nonarchimedean local field and let $\mathbf{G}$ be a Zariski-connected reductive k-group. For any element g of $\mathbf{G}(k)$ its scale $s(g)$ equals the product of the absolute values of those eigenvalues of $Ad(g)$ whose valuation is greater than 1 (counted with their proper multiplicities).*

Proof: Observe first that the claim will be true for an element g whenever it is true for a positive power of g. Likewise, using part (2) of Lemma 3.22, we see

that we may replace g by g' whenever $g^{-1}g'$ is a compact element commuting with g. If k has positive characteristic some positive power of g is semisimple. If the characteristic of k is 0, g has a Jordan–Chevalley decomposition $g = g_s g_u = g_u g_s$ with g_s, $g_u \in \mathbf{G}(k)$ where g_s is semisimple and g_u is unipotent. Since unipotents are compact elements, we may replace g by its semisimple part g_s. So, regardless of the characteristic, we may assume from the outset that g is semisimple.

Let $\mathbf{S}$ be the smallest torus of $\mathbf{G}$ containing g, that is, let $\mathbf{S}$ be the Zariski-closure of the group generated by g. From this characterization it is immediate that $\mathbf{S}(k)$ is Zariski-dense in $\mathbf{S}$. By the Galois-criterion $\mathbf{S}$ is defined over k. Its largest split, respectively anisotropic, tori $\mathbf{S}_d$ and $\mathbf{S}_a$ are defined over k as well.

Write g as $g'a$, where g' is in $\mathbf{S}_d(k)$ and a is in $\mathbf{S}_a(k)$. By our introductory remarks we may suppose that $g = g'$ is in fact contained in a k-split torus.

We are going to use the Remarque after Corollaire 3.18 in [BT65]. Let σ be the set of roots of $\Phi(\mathbf{S}_d, \mathbf{G})$ with $|\sigma(g)| > 1$. It is a closed (even connected) subset and by [Bou68, Chapitre VI,§1, Proposition 22] there is an ordering on $\Phi(\mathbf{S}_d, \mathbf{G})$ such that all the roots in σ are positive. Hence σ is unipotent. The set of k-rational points of the associated unipotent group $\mathbf{U}_\sigma$ is equal to $U_{g^{-1}}$, hence $U_{g^{-1}}$ is already closed. This follows from the fact that $\mathrm{Lie}(\mathbf{G})$ is the direct sum of $\mathrm{Lie}(\mathbf{U}_\sigma)$, $\mathrm{Lie}(\mathbf{U}_{-\sigma})$ and the sum of the eigenspaces of g to eigenvalues with valuation 1. In particular, each eigenspace of g to an eigenvalue with valuation greater than 1 is contained in $\mathrm{Lie}(\mathbf{U}_\sigma)$.

Arrange the elements $b_1, \ldots, b_m$ of σ in increasing order and put $\sigma_i := \{b_i, \ldots, b_m\}$. The group $\mathbf{U}_\sigma$ is defined and split over k. More precisely, $\mathbf{U}_\sigma = \mathbf{U}_{\sigma_1} \geqslant \cdots \geqslant \mathbf{U}_{\sigma_m}$ is a filtration by unipotent k-groups whose successive quotients admit a structure of k-vector space such that any element x of $\mathbf{S}_d$ acts by scalar multiplication with $b_i(x)$ on $\mathbf{U}_{\sigma_i}/\mathbf{U}_{\sigma_{i+1}}$. It follows that $\Delta_{U_{g^{-1}}}(g) = \Delta_{\mathbf{U}_\sigma(k)}(g) = \prod_{i=1}^m |b_i(g)|^{d_i}$, where $d_i = \dim(\mathbf{U}_{\sigma_i}(k)/\mathbf{U}_{\sigma_{i+1}}(k))$. This is exactly what we claimed. ∎

As a further result, contraction groups are usually unbounded.

PROPOSITION 3.24: *The following statements are equivalent:*

(1) $s_G(\alpha^{-1}) = 1$.
(2) U_α *is bounded.*
(3) $\forall O$ *tidy for* α: $O \subseteq P_{\alpha^{-1}}$.
(4) $\exists O$ *tidy for* α: $O \subseteq P_{\alpha^{-1}}$.
(5) $P_{\alpha^{-1}}$ *is open.*
(A) $U_\alpha \subseteq M_\alpha$.

(B) $M_\alpha = P_\alpha$.
(C) $\overline{U}_\alpha = U_0 := \overline{U}_\alpha \cap \overline{U}_{\alpha^{-1}}$.

Proof: We will first prove equivalence of the statements (1)–(5). The proof that the characterizations (A)–(C) are equivalent uses some results yet to come but are listed here for convenience of reference. These equivalences will not be used until later.

(2) $\Rightarrow$ (1): This is immediate combining parts (1) and (3) of Proposition 3.21 (with $H := \overline{U}_\alpha$). We prove that (1) implies (2) and (3). Take O tidy for α. The equality $1 = s_G(\alpha^{-1}) = |\alpha^{-1}(O) : \alpha^{-1}(O) \cap O|$ is equivalent to $O \supseteq \alpha^{-1}(O)$ and to $O_+ = O$. The first of these implies $O_{--} \subseteq \bigcup_{i\in\mathbb{N}} \alpha^{-i}(O) \subseteq O$, hence that $U_\alpha \subseteq O$ is bounded, giving (2). Further, $O = O_+ = O \cap P_{\alpha^{-1}}$ gives $O \subseteq P_{\alpha^{-1}}$, hence (3) follows.

Obviously (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Finally, to prove that (5) implies (1) take O tidy for α in $P_{\alpha^{-1}}$. Since we assume that $P_{\alpha^{-1}}$ is open, O is tidy for α in the ambient group G as well and we get $O_+ = O \cap P_{\alpha^{-1}} = O$ which is equivalent to $1 = s_G(\alpha^{-1})$ as has already been seen.

We now prove equivalence of (2) and the statements (A)–(C). If we assume (2) then using α-invariance of U_α we get that $\bigcup_{i\in\mathbb{Z}} \alpha^i(U_\alpha) = U_\alpha$ is bounded. The definition of M_α then gives $U_\alpha \subseteq M_\alpha$, that is (A).

The statements (A) and (B) are equivalent thanks to Corollary 3.17. For the remaining implications we need some results yet to be proved. If we assume (B), then since M_α is closed $\overline{U}_\alpha = \overline{U}_\alpha \cap M_\alpha$ which equals U_0 by Lemma 3.29 and we have derived (C). Assuming (C) we have U_α contained in U_0, which is a compact group by Corollary 3.27 giving (2). The proof is complete. ∎

3.4. SMALL TIDY SUBGROUPS. The contraction group, U_α, is closed if and only if there are arbitrarily small subgroups tidy for α. This and numerous other equivalences are established below in Theorem 3.32. Careful examination of the tidying procedure, Algorithm 2.2, is required for the proofs.

COROLLARY 3.25 (to Proposition 3.16): *If V is a tidy subgroup produced by the Algorithm 2.2 from the compact, open subgroup O, then $V_{--} = \overline{O_{--}}$.*

Proof: We go through the steps of the Algorithm 2.2. Step 1 produces a subgroup O' such that $O'_- = \alpha^n(O_-)$, hence

$$O'_{--} = O_{--}.$$

The group $L = \overline{\mathcal{L}}$ produced in step 2 of the algorithm equals $V_0 = O_0''$ by Lemma 3.8 in [Wil01] and is easily seen, by its definition, to be contained in $\overline{O'_{--}}$. Hence, by Proposition 3.16,

$$V_{--} = U_\alpha V_0 = U_\alpha L \leqslant U_\alpha \overline{O'_{--}} = \overline{O'_{--}} = \overline{O_{--}}.$$

To show the reverse inclusion we proceed to step 3 of the algorithm. Lemma 3.4 in [Wil01] implies

$$O^*_{--} = O'_{--}.$$

From $O^* \leqslant V$ we get immediately $O^*_{-} \leqslant V_-$ and $O^*_{--} \leqslant V_{--}$, hence

$$\overline{O_{--}} = \overline{O'_{--}} = \overline{O^*_{--}} \leqslant \overline{V_{--}} = V_{--}$$

since V is tidy. We are done. ∎

As an immediate consequence, we have the following characterisation of the closure of U_α.

THEOREM 3.26: $\overline{U}_\alpha = \bigcap\{V_{--} : V \text{ is tidy for } \alpha\}$.

Proof: We already know $U_\alpha \subseteq V_{--}$ for every compact open subgroup V. Since V_{--} is closed for V tidy, the inclusion $\overline{U}_\alpha \subseteq \bigcap\{V_{--} : V \text{ is tidy for } \alpha\}$ follows.

Now let $v \notin \overline{U}_\alpha$. We have to find a subgroup V tidy for α such that $v \notin V_{--}$. There is a compact open subgroup O such that $vO \cap \overline{U}_\alpha = \emptyset$. Then $vO \cap \overline{U}_\alpha O_0 = \emptyset$ and it follows that

$$v \notin \overline{U_\alpha O_0} = \overline{O_{--}}.$$

Thanks to Corollary 3.25, $v \notin V_{--}$, where V is the tidy subgroup constructed from O. We are done. ∎

One would expect the set of elements where α and α^{-1} are contracting to be small.

COROLLARY 3.27: *The group U_0, defined as $\overline{U}_\alpha \cap \overline{U}_{\alpha^{-1}}$, is equal to*

$$\bigcap\{V_0 : V \text{ is tidy for } \alpha\},$$

and hence to

$$\bigcap\{V : V \text{ is tidy for } \alpha\}$$

as well. In particular, it is compact.

Proof: First observe that if V is tidy, then so is $\alpha^n(V)$ for any integer n. Therefore

$$\bigcap\{V\colon V \text{ is tidy for } \alpha\} = \bigcap_{V \text{ tidy}} \bigcap_{n\in\mathbb{Z}} \alpha^n(V) = \bigcap\{V_0\colon V \text{ is tidy for } \alpha\}.$$

This is a compact group. It remains to show that it equals U_0.

For this, note that, by Theorem 3.26, U_0 is the intersection of all pairs V_{--} and W_{++} where V and W run through all tidy subgroups.

Since $V_0 \subseteq V_{--}$ and $W_0 \subseteq W_{++}$, it follows that

$$\bigcap\{V_0\colon V \text{ is tidy for } \alpha\} \subseteq U_0.$$

The next lemma implies the reverse inclusion.

LEMMA 3.28: *If V is a tidy subgroup of a totally disconnected locally compact group, then $V_{++} \cap V_{--} = V_0$.*

Proof: As already seen $V_0 = V_+ \cap V_- \subseteq V_{++} \cap V_{--}$ always. Conversely, if $x \in V_{++} \cap V_{--}$ then

$$x \in \alpha^k(V_+) \cap \alpha^l(V_-) \quad \text{for all } k \geq k_0 \geq 0 \geq l_0 \geq l.$$

Now $\alpha^{-k}(x) \in V_+ \subseteq V\ \forall k \geq k_0$ and $\alpha^{-l}(x) \in V_- \subseteq V\ \forall l \leq l_0$. Therefore $x \in \mathcal{L}$. Assuming V to be tidy gives $\mathcal{L} = V_0$, implying that $x \in V_0$. The proof of the corollary is complete. ∎ ∎

Considering Example 3.13(3) with $H = O$ acted on by the shift σ, we see that $U_\sigma \cap U_{\sigma^{-1}}$ is dense in $G = O^{\mathbb{Z}}$ and consequently $U_0 = G$. This example appears in connection with the investigation of semistable probability laws on G.

We will now turn to the question of existence of arbitrarily small tidy subgroups for an automorphism α. Corollary 3.27 shows that U_0 is an obstruction to their existence. We attempt now to make this more precise.

LEMMA 3.29: $\overline{U}_\alpha \cap P_{\alpha^{-1}} = \overline{U}_\alpha \cap M_\alpha = U_0$.

Proof: The proof is an easy direct computation using Theorem 3.26 and that

$V_0 = V \cap M_\alpha$ for tidy V:

$$U_0 = \overline{U}_\alpha \cap \overline{U}_{\alpha^{-1}} \subseteq \overline{U}_\alpha \cap P_{\alpha^{-1}} = \overline{U}_\alpha \cap M_\alpha = \bigcap_{V \text{ tidy}} V_{--} \cap M_\alpha$$
$$= \bigcap_{V \text{ tidy}} (V_{--} \cap M_\alpha) = \bigcap_{V \text{ tidy}} (\bigcup_{i \in \mathbb{N}} \alpha^{-i}(V \cap P_\alpha) \cap M_\alpha)$$
$$= \bigcap_{V \text{ tidy}} \bigcup_{i \in \mathbb{N}} \alpha^{-i}(V \cap M_\alpha) = \bigcap_{V \text{ tidy}} \alpha^{-i}(V_0) = \bigcap_{V \text{ tidy}} V_0 = U_0. \quad \blacksquare$$

As a Corollary we get a nice factorization of $\overline{U}_\alpha$.

COROLLARY 3.30: *$\overline{U}_\alpha = U_0 U_\alpha$, In other words $\overline{U}_\alpha = U_{\alpha/U_0}$.*

Proof: Let α_1 be the restriction of α to $\overline{U}_\alpha$. Then $U_{\alpha_1} = U_\alpha$, $M_{\alpha_1} = U_0$ by Lemma 3.29 and $P_{\alpha_1} = \overline{U}_\alpha$ and the claim follows by Corollary 3.17. $\blacksquare$

After a further lemma we will be able to characterize the automorphisms with arbitrarily small tidy subgroups. Section 2 of [Wil] describes a tidying procedure which differs from Algorithm 2.2 by taking in step 3 the group $K := \overline{U_\alpha \cap P_{\alpha^{-1}}} \subseteq U_0$ instead of L (constructed in step 2, which becomes superfluous). That is, we now put $O^* := \{x \in O' \colon kxk^{-1} \in O'K \ \forall k \in K\}$ and $O'' := O^*K$. According to Lemma 3.3(1) in [Wil], O^* is a compact open subgroup. It is smaller than O. We work with this algorithm in the proof of the lemma below.

It is proved in *loc. cit.* that the outputs of the two algorithms are the same. One may wonder whether the reason for this is that $K = U_0$. It follows from (2) and (1) below that this is indeed the case.

LEMMA 3.31:

(1) $U_\alpha \cap P_{\alpha^{-1}} = U_\alpha \cap M_\alpha = U_\alpha \cap U_0$.

(2) $\overline{U_\alpha \cap M_\alpha} = U_0$; *in particular, U_0 is the unique tidy subgroup for α restricted to U_0.*

(3) *A compact, open subgroup is tidy if and only if it satisfies (T1) and contains U_0.*

Proof: The first statement is easily derived from the descriptions of U_0 in Lemma 3.29.

From (1) we get that $\overline{U_\alpha \cap M_\alpha}$ equals the group K introduced above. It follows from Lemma 3.29 that $K \subseteq U_0$, so assume we are given an element x not in K. Since K is closed, there is a compact open subgroup O such that $xO \cap K$ is empty. In other words, x does not belong to KO.

Applying the modified algorithm to O, we have $KO \supseteq KO^* = O''$ is a tidy subgroup, and $x \notin KO$ implies that x is not in U_0 by Corollary 3.27. Thus $U_0 \subseteq K$ and therefore $U_0 = K = \overline{U_\alpha \cap M_\alpha}$ as claimed in (2).

To see the remaining part of sub-claim (2) observe that the contraction group of the restriction of α to U_0 is $U_\alpha \cap U_0$ and similarly for α^{-1}. Then, by what has already been shown, $\overline{U_\alpha \cap U_0} \cap \overline{U_{\alpha^{-1}} \cap U_0} = \overline{U_\alpha \cap M_\alpha} \cap \overline{U_{\alpha^{-1}} \cap M_\alpha} = U_0 \cap U_0 = U_0$ and there can be no proper subgroup of U_0 tidy for the restriction of α to U_0 by Corollary 3.27.

We turn to (3). The conditions are clearly necessary for a subgroup O to be tidy. If they are satisfied, the variant of the tidying-up procedure described above leaves the compact open subgroup unchanged. This shows that O is tidy. ∎

In all examples examined so far, U_0 is equal to $\overline{U_\alpha \cap U_{\alpha^{-1}}}$. We do not know whether this holds in general.

The existence of arbitrarily small tidy subgroups for a given automorphism has many equivalent reformulations.

THEOREM 3.32: *The following conditions are equivalent:*

(1) U_α *is closed.*
(1′) $U_{\alpha^{-1}}$ *is closed.*
(2) $U_\alpha \cap M_\alpha = 1$.
(2′) $U_{\alpha^{-1}} \cap M_\alpha = 1$.
(3) $\overline{U}_\alpha \cap M_\alpha = 1$.
(3′) $\overline{U}_{\alpha^{-1}} \cap M_\alpha = 1$.
(4) $U_0 = 1$.
(4′) *There are arbitrarily small tidy subgroups for* α.
(5) *For all compact, open* $O \leqslant G$ *there is a* k *such that* ${}^kO$ *is tidy.*
(5′) *A compact, open subgroup satisfying (T1) satisfies (T2) as well.*
(6) $P_\alpha = M_\alpha \ltimes U_\alpha$ *topologically.*
(6′) $P_{\alpha^{-1}} = M_\alpha \ltimes U_{\alpha^{-1}}$ *topologically.*

Proof: (5′) $\iff$ (5): Assume (5′). Run step 1 of Algorithm 2.2 on the given O to obtain another compact, open subgroup of the stated form satisfying (T1) hence (T2) by assumption. Conversely, assume that O satisfies (T1). By assumption $O' := \bigcap_{i=0}^{k} \alpha^i(O)$ is tidy. As already used several times $O'_+ = O_+$ and $O'_- = \alpha^k(O_-)$, hence $O_{++} = O'_{++}$ and $O_{--} = O'_{--}$ are closed since O' is tidy. Hence O is tidy and (T1) is shown to imply (T2).

Since the procedure applied to O in (5) shrinks the compact, open subgroups in question, we have that (5) $\Rightarrow$ (4'), which evidently implies (4). Lemma 3.29 shows that (4), (3) and (3') are all equivalent.

If we assume that U_α is not closed, then $1 \neq U_0$ by Corollary 3.30. Hence $M_\alpha \cap \overline{U}_\alpha \neq 1$ and we have shown that (3) implies (1). By symmetry, (3') implies (1') as well.

Assuming (1) we find for any compact, open subgroup V that $V_{--} = V_0 U_\alpha$ is closed, since V_0 is compact. Assuming V satisfies (T1) we get from [Wil94, Lemma 3(b)] that V_{++} is closed as well. This means that property (T2) is automatic once (T1) is known to hold, i.e. (5'). By symmetry again (1') implies (5'). So far we have established equivalence of all conditions listed except (2), (2'), (6) and (6').

It is trivial that (3) implies (2) and (3') implies (2'). Lemma 3.31(2) gives that (2) implies (4). Symmetrically (2') implies (4). Together with (4) $\iff$ (3), this leaves to prove the equivalence of (6) and (6') with the other conditions:

By symmetry it suffices to prove that (6) is equivalent to (2). Property (6) evidently implies (2). Assume (2) to conclude that P_α is the semidirect product of M_α and U_α as an abstract group. We may then apply Proposition 6.17 from [RD81]. Our claim is that (6) holds, which is statement (d) in that Proposition. It is shown there that it is equivalent to the statement (b) that the map $M_\alpha \to P_\alpha / U_\alpha$ obtained by restriction of the quotient map modulo the normal subgroup U_α is a topological isomorphism. Since we know that (2) implies (1), this map is the one considered in Lemma 3.18, where we showed that it is an identification in general. Under the assumption of (2) however, the kernel of this map is trivial and (b), thus (6), is established. We are done. ∎

We now list some examples where the conditions of Theorem 3.32 hold.

Remark 3.33: All contraction groups for general/inner automorphisms are closed in the following cases.

(1) Groups with trivial contraction groups, among them:
   (a) discrete groups, with respect to **all** and
   (b) SIN-groups with respect to **inner** automorphisms.
   (c) MAP-groups with respect to **inner** automorphisms. (The von-Neumann-kernel of a locally compact group contains all contraction groups of inner automorphisms: Let ρ be a continuous finite-dimensional unitary representation of the group. Consider an element, x, say. Since $\rho(x)$ is unitary, the corresponding eigenvalues have absolute value 1. A trivial modification of Lemma II.(3.2) from

[Mar89] shows that U_x fixes each eigenvector of $\rho(x)$. Since there is a basis of the representation space consisting of eigenvectors of $\rho(x)$, we are done.)

(d) nilpotent-by-compact groups. These are distal by [Ros79, Proposition 3]. This, by its definition (*loc. cit.*, introduction), easily implies that all contraction groups of **inner** automorphisms are trivial.

(2) Any (totally disconnected) locally compact group having an open subgroup satisfying the ascending chain condition on its closed subgroups. Thanks to ([Wan84, Lemma 3.2]) the group then satisfies the condition (1) of Theorem 3.32 with respect to **any** automorphism ([Wan84, Lemma 3.2]). This criterion applies to any p-adic Lie group ([Wan84, Theorem 3.5]), hence to any analytic group over any nonarchimedian field of characteristic 0.

(3) Any closed subgroup of a linear group over a local field with respect to **inner** automorphisms. By the introductory remark it is enough to prove this for the group SL_n, since we may embed any closed linear group therein as a closed subgroup. We have already seen in Example 3.13(1) that a contraction subgroup for an element of SL_n is an algebraic subgroup, hence is closed.

Further examples with all contraction groups for inner automorphisms closed may be obtained from the above list by forming projective limits and restricted products.

Remark 3.34:

(1) Note that item 1(c) above shows that all totally disconnected locally compact MAP-groups have trivial inner contraction groups, hence scale function identically one (they are **uniscalar**) if they are metric. This is an improvement over the main result of [LR68].

(2) Let G^+ be the closed subgroup generated by all contraction groups of inner automorphisms. It is stable under any (bicontinuous) automorphism of G, hence is in particular normal. As a consequence of Proposition 3.7, for metric groups G, G/G^+ is uniscalar. The scale function should thus characterize best the groups satisfying $G = G^+$. Every non-uniscalar topologically simple group will belong to this class by Proposition 3.24, but no solvable group will because $G^+ \subseteq \overline{[G,G]}$. This last remark suggests that we define subgroups G^{n+} by iterating the definition of G^+.

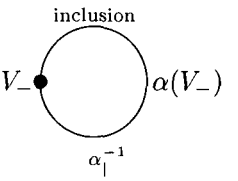
4. The tree-representation theorem

We will show that, if V is tidy for α, then V_{--} extended by the cyclic group generated by α, that is, $V_{--} \rtimes \langle\alpha\rangle$, has a representation onto some closed subgroup of the automorphism group of a homogeneous tree. This representation can be used to analyse groups of this type, which are the simplest non-uniscalar groups.

Although this is not the approach we shall take, this representation is an instance of a general construction in the Bass–Serre theory of group actions on graphs which is described in [Ser80] and [DD89]. Since α: $V_- \to \alpha(V_-)$ is an injection into V_-, the group $V_{--} \rtimes \langle\alpha\rangle$ is isomorphic to the HNN extension

$$V_- \underset{\alpha(V_-)}{*} t, \quad \text{where } t = \alpha^{-1}\colon \alpha(V_-) \to V_-.$$

The HNN extension is defined in [DD89, Example 3.5(v)] to be the fundamental group of the graph of groups at right. It is shown in [Ser80] and [DD89] how to represent HNN extensions on trees. For completeness, we shall describe the tree and the action of $V_{--} \rtimes \langle\alpha\rangle$ directly. Elements of $V_{--} \rtimes \langle\alpha\rangle$ will be denoted $v\alpha^m$.



The tree will be denoted by T. Its vertices are the left V_--cosets in $V_{--} \rtimes \langle\alpha\rangle$. Distinct vertices xV_- and yV_- are linked by a directed edge $xV_- \to yV_-$ if and only if

$$yV_- \subset xV_-\alpha = (x\alpha)\alpha^{-1}(V_-). \tag{1}$$

Equivalently, there is an edge from $v\alpha^m V_-$ to $w\alpha^n V_-$ if and only if $n = m + 1$ and $w \in v\alpha^m(V_-)$. Since $\alpha^{-1}(V_-)$ is the union of $s(\alpha^{-1})$ V_--cosets, there are $s(\alpha^{-1})$ out-edges from the vertex xV_-. Since $xV_- \subset \big((x\alpha^{-1})\alpha\big)\,\alpha^{-1}(V_-)$ there is one edge into xV_-, from the vertex $(x\alpha^{-1})V_-$. Hence each vertex in T has degree $s(\alpha^{-1}) + 1$.

We show next that T is a tree when α has infinite order. For each $n \in \mathbb{Z}$, denote the vertex $\alpha^n V_-$ by $V^{(n)}$. Then $V^{(n)} = \alpha^{n-1}\alpha(V_-)\alpha \subset \alpha^{n-1}V_-\alpha$ and it follows from (1) that

$$\ldots, V^{(-2)}, V^{(-1)}, V^{(0)}, V^{(1)}, V^{(2)}, \ldots$$

is an infinite path, call it P, in T. Consider a general vertex $v\alpha^m V_-$ in T, where $v \in \alpha^{-n}(V_-)$. Then $v\alpha^m V_- = \alpha^m w V_-$, where $w = \alpha^{-m}(v) \in \alpha^{-(m+n)}(V_-)$.

If $m+n \leq 0$, it follows that $v\alpha^m V_- = V^{(m)}$. If $m+n > 0$, then $v\alpha^m V_-$ is descended by a path of length $m+n$ from $V^{(-n)}$. In either case we see that $v\alpha^m V_-$ is connected to P. Therefore T is connected. To see that there are no circuits in T, observe first of all that, since each vertex has only one in-edge, any circuit without backtracking must be a directed path. However, the power of α in each coset xV_- strictly increases in the direction of any path in T and so there can be no directed circuit unless α has finite order.

The action of $V_{--} \rtimes \langle\alpha\rangle$ on the vertices of T is the usual action of the group on a space of left V_--cosets. It is clear that this action preserves the adjacency relation defined in (1). Denote this action by ρ: $V_{--} \rtimes \langle\alpha\rangle \to \mathrm{Aut}(T)$. Then ρ is vertex and edge transitive on T and so the quotient graph is a loop.

Denote the set of ends of T by ∂T, the end of the path P corresponding to $\{V^{(n)}\}_{n=1}^{\infty}$ by ∞, and the other end of P by $-\infty$. Then α acts as a translation of distance 1 on P with $-\infty$ as the repelling end. Since every path in P descends ultimately from $-\infty$ and it is the unique end of T having this property, $-\infty$ is fixed by $\rho(V_{--} \rtimes \langle\alpha\rangle)$. This may also be shown directly by verifying that $v\alpha^m V^{(-r)} = V^{(m-r)}$ provided that r is sufficiently large that $\alpha^{r-m}(v) \in V_-$.

Recalling that the automorphism group of T is itself a totally disconnected locally compact group when equipped with the topology of uniform convergence on compact sets, we are in a position to state the tree-representation theorem.

THEOREM 4.1: *Let G be a totally disconnected locally compact group, α an automorphism of G of infinite order and let V be tidy for α. Then $V_{--} \rtimes \langle\alpha\rangle \to \mathrm{Aut}(T)$ is a continuous representation ρ onto a closed subgroup of the automorphism group of a homogeneous tree T of degree $s(\alpha^{-1})+1$.*

(1) *The action of $V_{--} \rtimes \langle\alpha\rangle$ on $\mathrm{Aut}(T)$: fixes an end, $-\infty$; is transitive on $\partial T \setminus \{-\infty\}$; and the quotient graph is a loop.*

(2) *The stabiliser of each end in $\partial T \setminus \{-\infty\}$ is a conjugate of $V_0 \rtimes \langle\alpha\rangle$. The kernel of ρ is the largest compact normal α-stable subgroup of V_{--}.*

(3) *The image of V_{--} under ρ is the set of elliptic elements in $\rho(V_{--} \rtimes \langle\alpha\rangle)$.*

Proof: Stabilisers of vertices xV_- in T form a subbasis for the topology on $\mathrm{Aut}(T)$. Since $\rho^{-1}(\mathrm{stab}(xV_-)) = xV_-x^{-1}$, which is open in $V_{--} \rtimes \langle\alpha\rangle$, it follows that ρ is continuous. The continuity of ρ and the compactness of V_- imply that $\rho(V_{--} \rtimes \langle\alpha\rangle) \cap \mathrm{stab}(V^{(0)}) = \rho(V_-)$ is compact and hence closed. Therefore $\rho(V_{--} \rtimes \langle\alpha\rangle)$ is closed by [HR79, II.(5.9)].

(1) It remains only to show that ρ is transitive on $\partial T \setminus \{-\infty\}$ and, for this, it suffices to show that for each $\omega \in \partial T \setminus \{-\infty\}$ there is $v \in V_{--}$ such that $v \cdot \infty =$

ω. Let $\{v_n\alpha^n V_-\}_{n=1}^{\infty}$ be a path converging to ω. Since $(v_n\alpha^n V_-, v_{n+1}\alpha^{n+1}V_-)$ is an edge, it follows from (1) that $v_{n+1}\alpha^{n+1}(V_-) \subset v_n\alpha^n(V_-)$ for each n. Each of these sets is compact and so $\bigcap_{n=1}^{\infty} v_n\alpha^n(V_-) \neq \emptyset$. Choose v in this intersection, so that $v\alpha^n(V_-) = v_n\alpha^n(V_-)$ for each n. Then

$$vV^{(n)} = v\alpha^n V_- = v\alpha^n(V_-)\alpha^n = v_n\alpha^n(V_-)\alpha^n = v_n\alpha^n V_-$$

for each n and it follows that $v \cdot \infty = \omega$.

(2) For the first part, it suffices to show that the stabiliser of ∞ equals $V_0 \rtimes \langle\alpha\rangle$. It is clear that this group is contained in the stabiliser. Conversely, if x stabilises ∞, then it leaves the path P invariant and in fact translates P by a distance, d say. Then $x\alpha^{-d}$ fixes every vertex on P. Hence $x\alpha^{-d} \in \bigcap_{n\in\mathbb{Z}} \alpha^n(V_-) = V_0$ and we have $x \in V_0 \rtimes \langle\alpha\rangle$.

The kernel of ρ is the intersection of all the vertex stabilisers and is therefore a compact normal α-stable subgroup of V_{--}. If M is any such subgroup, then it fixes a point, p say, in T because it is compact. Since M is also normal in V_{--} and α-invariant, it fixes every point in the $V_{--} \rtimes \langle\alpha\rangle$-orbit of p. Since ρ is edge-transitive, it follows that M fixes every point in T and so M is contained in the kernel of ρ.

(3) Since the image of ρ fixes $-\infty$, if $g \in \mathrm{im}(\rho)$ fixes a point p, then it fixes every vertex on the path from $-\infty$ to p. Hence g fixes a point if and only if there is a $k \in \mathbb{Z}$ such that $g.V^{(k)} = V^{(k)}$. The set of elliptics therefore coincides with

$$\bigcup_{k\in\mathbb{Z}} \mathrm{stab}(V^{(k)}) \cap \mathrm{im}(\rho) = \bigcup_{k\in\mathbb{Z}} \rho(\alpha^k(V_-)) = \rho(V_{--}). \qquad \blacksquare$$

The representation ρ restricts to give a representation of $\overline{U}_\alpha \rtimes \langle\alpha\rangle$ on T. It follows from Theorem 3.16 that, for V_{--} metric, this is the same representation as obtained if Theorem 4.1 is applied with G equal to $\overline{U}_\alpha$. Hence all the assertions of Theorem 4.1 hold with V_{--} replaced by $\overline{U}_\alpha$. However, more can be said about the representation of $\overline{U}_\alpha \rtimes \langle\alpha\rangle$.

THEOREM 4.2: *Let G be a totally disconnected locally compact metric group and α an automorphism of G of infinite order. Let ρ be the representation of $\overline{U}_\alpha \rtimes \langle\alpha\rangle$ on the tree T as defined above.*

(1) *The action of U_α is transitive on $\partial T \setminus \{-\infty\}$ and is simply transitive if and only if U_α is closed.*

(2) *If U_α is closed, then ρ is a topological isomorphism onto its image.*

Proof: (1) Theorem 4.1(1) shows that $\overline{U}_\alpha$ is transtitive on the set $\partial T \setminus \{-\infty\}$. Since, by Corollary 3.30, $\overline{U}_\alpha = U_\alpha U_0$ and since U_0 is the stabiliser of ∞, we

have

$$\partial T \setminus \{-\infty\} = \overline{U}_\alpha \cdot \infty = U_\alpha \cdot \infty.$$

The action is simply transitive if and only if $\mathrm{stab}(\infty) = U_0 \cap U_\alpha$ is trivial. Since $U_0 \cap U_\alpha$ is dense in U_0, Theorem 3.32 shows that $U_0 \cap U_\alpha$ is trivial if and only if U_α is closed.

(2) If U_α is closed, then α is compactly contractive. Hence the compact kernel of ρ is trivial and ρ is faithful. ∎

Subsection 3.2 examined the difference between the groups V_{--} and U_α where α is respectively shrinking and contracting. Both groups act on T with the U_α-action being the restriction of the V_{--}-action. The next example shows that the difference between the groups may, or may not, be seen in the action on the tree.

Example 4.3: (1) Let G be the group of upper triangular matrices in $SL_2(\mathbb{Q}_p)$. The contraction group with respect to the automorphism α given by inner conjugation by $\mathrm{diag}(p, p^{-1})$ is the group of strict upper triangular matrices. The subgroup of upper triangular matrices with entries in $\mathbb{Z}_p$ is compact open in G and satisfies $\alpha(V) \subseteq V$, hence is tidy by Example 2.3. The group V_{--} is then the subgroup of elements in G having diagonal entries in $\mathbb{Z}_p^*$. The tree representation ρ of V_{--} is seen to be the composite of the inclusion in $SL_2(\mathbb{Q}_p)$ with the representation of $SL_2(\mathbb{Q}_p)$ on its Bruhat–Tits tree. Since V_{--} properly contains U_α, ρ distinguishes between these groups.

(2) On the other hand, if we are given a closed contraction group U_α we may construct a strictly larger group G, an extension of α to G and a tidy subgroup V for α acting on G with $G = V_{--}$ such that the tree representation ρ defined by V_- satisfies $\rho(V_{--}) = \rho(U_\alpha)$ as follows. Take any nontrivial compact group K and define G by $G := K \times U_\alpha$. Let α act trivially on K. Choose a tidy subgroup O for α in U_α. Then $V := K \times O$ is tidy for α in G and $V_{--} = K \times O_{--} = K \times U_\alpha = G$. Obviously $\rho(V_{--}) = \rho(U_\alpha)$.

Part (2) of the example has V_{--} as the direct product of U_α and the kernel of ρ. That is not special to this example. The kernel of ρ and $\overline{U}_\alpha$ are both closed normal subgroups of V_{--} and so their product is always a subgroup and it is closed because $\ker \rho$ is compact. Group isomorphism theorems plus the fact that, if U_α is closed, then $U_\alpha \cap \ker \rho$ is trivial imply the following result.

PROPOSITION 4.4: *Let V be tidy for the automorphism α of the totally disconnected locally compact metric group G.*

(1) *If $\rho(\overline{U}_\alpha) = \rho(V_{--})$, then $V_{--} = \overline{U}_\alpha \ker \rho$.*

(2) *If, furthermore, U_α is closed, then $V_{--} = U_\alpha \times \ker \rho$.*

Remark 4.5: Essentially the same tree as constructed here has been associated with tidy subgroups and contraction groups elsewhere. In Theorem 3.4 of [Möl02] a tree construction is used to translate the definition of tidy subgroup into permutation group theoretic terms. The rooted tree constructed there is a branch of the tree constructed above. Also, the underlying tree constructed in Theorem 4.1, but not the group representation ρ, is already implicit in Proposition 3.7 in [Sie86], which studies the case where the contraction group is closed.

References

[Bou68] N. Bourbaki, *Groupes et Algèbres de Lie, Chapitres 4, 5 et 6,* Hermann, Paris, 1968; publiées dans Actualités scientifiques et industriels 1337.

[BT65] A. Borel and J. Tits, *Groupes réductifs,* Publications mathématiques de l'Institut des Hautes Études Scientifiques **27** (1965), 55–150.

[DD89] W. Dicks and M. J. Dunwoody, *Groups acting on Graphs,* Volume 17 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1989.

[DS91] S. G. Dani and R. Shah, *Contraction subgroups and semistable measures on p-adic Lie groups,* Mathematical Proceedings of the Cambridge Philosophical Society **110** (1991), 299–306.

[Glö98a] H. Glöckner, *Scale functions on linear groups over local skew fields,* Journal of Algebra **205** (1998), 525–541.

[Glö98b] H. Glöckner, *Scale functions on p-adic Lie groups,* Manuscripta Mathematica **97** (1998), 205–215.

[GW02] H. Glöckner and G. A. Willis, *Topologization of Hecke pairs and Hecke C^*-algebras,* Topology Proceedings **26** (2001–2002), 565–591.

[Haz88] M. Hazewinkel (ed.), *Encyclopaedia of Mathematics. Vol. 1. A–B,* D. Reidel Publ. Co., Dordrecht, 1988. Translated from the Russian.

[HR79] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis; Volume I,* Volume 115 of Grundlehren der mathematischen Wissenschaften, second edition, Springer-Verlag, Berlin, 1979.

[HS88] W. Hazod and E. Siebert, *Automorphisms on a Lie group contracting modulo a compact subgroup and applications to semistable convolution semigroups,* Journal of Theoretical Probability **1** (1988), 211–225.

[LR68] H. Leptin and L. Robertson, *Every locally compact MAP group is unimodular,* Proceedings of the American Mathematical Society **19** (1968), 1079–1082.

[Mar89] G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Volume 17 (3. Folge) of Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, New York, 1989.

[Möl02] R. G. Möller, *Structure theory of totally disconnected locally compact groups via graphs and permutations*, Canadian Journal of Mathematics **54** (2002), 795–827.

[Moo80] C. C. Moore, *The Mautner phenomenon for general unitary representations*, Pacific Journal of Mathematics **86** (1980), 155–169.

[MR76] P. R. Müller-Römer, *Kontrahierende Erweiterungen und kontrahierbare Gruppen*, Journal für die reine und angewandte Mathematik **283/284** (1976), 238–264.

[Pra82] G. Prasad, *Elementary proof of a theorem of Bruhat–Tits–Rousseau and of a theorem of Tits*, Bulletin de la Société Mathématique de France **110** (1982), 197–202.

[RD81] W. Roelcke and S. Dierolf, *Uniform Structures on Topological Groups and their Quotients*, McGraw-Hill International Book Company, New York, 1981.

[Ros79] J. Rosenblatt, *A distal property of groups and the growth of connected locally compact groups*, Mathematika **26** (1979), 94–98.

[Ser80] J.-P. Serre, *Trees*, Springer-Verlag, Berlin, 1980. The original French edition *Arbres, amalgames*, SL_2 was published as Astérisque **46** (by Société Mathématique de France) in 1977.

[Sie86] E. Siebert, *Contractive automorphisms on locally compact groups*, Mathematische Zeitschrift **191** (1986), 73–90.

[Sie89] E. Siebert, *Semistable convolution semigroups and the topology of contraction groups*, in *Probability Measures on Groups IX (Proceedings of a Conference held at Oberwolfach/FRG 1988)*, Lecture Notes in Mathematics **1379**, Springer-Verlag, Berlin, 1989, pp. 325–343.

[Wan84] J. S. P. Wang, *The Mautner phenomenon for p-adic Lie groups*, Mathematische Zeitschrift **185** (1984), 403–412.

[Wil] G. A. Willis, *Tidy subgroups for commuting automorphisms of totally disconnected locally compact groups: An analogue of simultaneous triangularisation of matrices*, New York Journal of Mathematics **10** (2004), 1–35. Available at http://nyjm.albany.edu:8000/j/2004/Vol10.htm.

[Wil94] G. A. Willis, *The structure of totally disconnected locally compact groups*, Mathematische Annalen **300** (1994), 341–363.

[Wil01] G. A. Willis, *Further properties of the scale function on a totally disconnected locally compact group*, Journal of Algebra **237** (2001), 142–164.